Занятие 3 Условия Коши-Римана

1) 177 $\rho=\left(x^2+y^2\right)e^y$.

$$\frac{\partial\rho}{\partial x}=2xe^y,\ \frac{\partial\rho}{\partial y}=\left(2y+x^2+y^2\right)e^y$$

$$\begin{cases}\dfrac{\partial\theta}{\partial x}=-\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial y}\\ \dfrac{\partial\theta}{\partial y}=\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial x}\end{cases}$$

$$\begin{cases}\dfrac{\partial\theta}{\partial x}=-\dfrac{2y+x^2+y^2}{x^2+y^2}=-\dfrac{2y}{x^2+y^2}-1\\ \dfrac{\partial\theta}{\partial y}=\dfrac{2x}{x^2+y^2}\end{cases}$$

$$\theta=2\operatorname{arctg}\frac{y}{x}-x$$

$$w=\rho e^{i\theta}=\left(x^2+y^2\right)e^y e^{2i\operatorname{arctg}\frac{y}{x}}e^{-\iota x}=$$

$$=r^2e^{2i\operatorname{arctg}\frac{y}{x}}e^{-i(x+iy)}=z^2e^{-z}$$

Или

$$\left(x^2+y^2\right)e^{2i\operatorname{arctg}\frac{y}{x}}=\left(x^2+y^2\right)\left(\cos\left(2\operatorname{arctg}\frac{y}{x}\right)+i\sin\left(2\operatorname{arctg}\frac{y}{x}\right)\right)=$$

$$=\left(x^2+y^2\right)\left(\frac{1-\frac{y^2}{x^2}}{1+\frac{y^2}{x^2}}+i\frac{2\frac{y}{x}}{1+\frac{y^2}{x^2}}\right)=\left(x^2+y^2\right)\frac{x^2-y^2+2ixy}{x^2+y^2}=$$

$$=\left(x+iy\right)^2=z^2$$

2) $\rho=\operatorname{ch}2x+\cos 2y=2\operatorname{ch}^2x-2\sin^2y$

$$\begin{cases}\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial x}=\dfrac{\partial\theta}{\partial y}\\ -\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial y}=\dfrac{\partial\theta}{\partial x}\end{cases}$$

$$\frac{\partial\rho}{\partial x}=2\operatorname{sh}2x$$
$$\frac{\partial\rho}{\partial y}=-2\sin 2y$$

$$\begin{cases}\dfrac{\partial\theta}{\partial x}=\dfrac{2\sin 2y}{\operatorname{ch}2x+\cos 2y}\\ \dfrac{\partial\theta}{\partial y}=\dfrac{2\operatorname{sh}2x}{\operatorname{ch}2x+\cos 2y}\end{cases}$$

$$\theta=\int\frac{2\operatorname{sh}2x}{\operatorname{ch}2x+\cos 2y}dy=\left[u=\operatorname{tg}x\right]=2\operatorname{sh}2x\int\frac{1}{\operatorname{ch}2x+\frac{1-u^2}{1+u^2}}\frac{du}{1+u^2}+\psi\left(x\right)=$$

$$=2\operatorname{sh}2x\int\frac{1}{\left(1+u^2\right)\operatorname{ch}2x+\left(1-u^2\right)}du+\psi\left(x\right)=2\sin 2y\int\frac{1}{\left(\operatorname{ch}2x+1\right)+u^2\left(\operatorname{ch}2x-1\right)}du+\psi\left(x\right)=$$

$$=\operatorname{sh}2x\int\frac{1}{\operatorname{ch}^2x+u^2\operatorname{sh}^2x}du+\psi\left(x\right)=\frac{\operatorname{sh}2x}{\operatorname{sh}x\operatorname{ch}x}\operatorname{arctg}\left(u\operatorname{th}x\right)+\psi\left(x\right)=2\operatorname{arctg}\left(\operatorname{tg}y\operatorname{th}x\right)+\psi\left(x\right)$$

$$\frac{\partial\theta}{\partial x}=\frac{2\operatorname{tg}y}{\left(1+\operatorname{tg}^2y\operatorname{th}^2x\right)\operatorname{ch}^2x}+\psi'\left(x\right)=\frac{\sin 2y}{\operatorname{ch}^2x\cos^2y+\operatorname{sh}^2x\sin^2y}+\psi'\left(x\right)=$$

$$=\frac{4\sin 2y}{\left(\operatorname{ch}2x+1\right)\left(1+\cos 2y\right)+\left(\operatorname{ch}2x-1\right)\left(1-\cos 2y\right)}+\psi'\left(x\right)=\frac{2\sin 2y}{\operatorname{ch}2x+\cos 2y}+\psi'\left(x\right)$$

$$\theta=2\operatorname{arctg}\left(\operatorname{tg}y\operatorname{th}x\right)+C_1$$

$$w=\rho e^{i\theta}=C\cdot\left(\operatorname{ch}2x+\cos 2y\right)e^{2i\operatorname{arctg}\left(\operatorname{th}x\operatorname{tg}y\right)}=$$

$$=C\cdot\left(\operatorname{ch}2x+\cos 2y\right)\left(\cos\left(2\operatorname{arctg}\left(\operatorname{th}x\operatorname{tg}y\right)\right)+i\sin\left(2\operatorname{arctg}\left(\operatorname{th}x\operatorname{tg}y\right)\right)\right)=$$

$$=C\cdot\left(\operatorname{ch}2x+\cos 2y\right)\left(\frac{1-\operatorname{th}^2x\operatorname{tg}^2y+2i\operatorname{th}x\operatorname{tg}y}{1+\operatorname{th}^2x\operatorname{tg}^2y}\right)=$$

$$=C\cdot\left(\operatorname{ch}2x+\cos 2y\right)\frac{\operatorname{ch}^2x\cos^2y-\operatorname{sh}^2x\sin^2y+2i\operatorname{sh}x\operatorname{ch}x\sin y\cos y}{\operatorname{ch}^2x\cos^2y+\operatorname{sh}^2x\sin^2y}=$$

$$=C\cdot\left(\operatorname{ch}2x+\cos 2y\right)\frac{\left(\operatorname{ch}2x+1\right)\left(1+\cos 2y\right)-\left(\operatorname{ch}2x-1\right)\left(1-\cos 2y\right)+2i\operatorname{sh}2x\sin 2y}{\left(\operatorname{ch}2x+1\right)\left(1+\cos 2y\right)+\left(\operatorname{ch}2x-1\right)\left(1-\cos 2y\right)}=$$

$$=C\cdot\left(\operatorname{ch}2x+\cos 2y\right)\frac{2+2\operatorname{ch}2x\cos 2y+2i\operatorname{sh}2x\sin 2y}{2\operatorname{ch}2x+2\cos 2y}=C\left(1+\operatorname{ch}2x\cos 2y+i\operatorname{sh}2x\sin 2y\right)=$$

$$=C\cdot\left(1+\cos 2xi\cos 2y+i\sin 2xi\sin 2y\right)=C\cdot\left(1+\cos\left(2xi-2y\right)\right)=C\cdot\left(1+\cos\left(2i\left(x+iy\right)\right)\right)=$$

$$=C\cdot\left(1+\cos\left(2zi\right)\right)=C\cdot\left(1+\operatorname{ch}\left(2z\right)\right)=2C\cdot\operatorname{ch}^2z,\ \left|C\right|=1$$

3) 180 $\theta=\varphi+r\sin\varphi$.

Запишем условия Коши-Римана и полярных координатах:

$$f\left(re^{i\varphi}\right)=\rho\left(r,\varphi\right)e^{i\theta(r,j)};$$

$$f'\left(re^{i\varphi}\right)e^{i\varphi}=\frac{\partial\rho}{\partial r}e^{i\theta}+i\rho e^{i\theta}\frac{\partial\theta}{\partial r}\,|\,ri$$

$$f'\left(re^{i\varphi}\right)re^{i\varphi}i=\frac{\partial\rho}{\partial\varphi}e^{i\theta}+ire^{i\theta}\frac{\partial\theta}{\partial\varphi}$$

$$\frac{\partial\rho}{\partial r}e^{i\theta}ri-r\rho e^{i\theta}\frac{\partial\theta}{\partial r}=\frac{\partial\rho}{\partial\varphi}e^{i\theta}+ire^{i\theta}\frac{\partial\theta}{\partial\varphi}$$

$$\frac{\partial\rho}{\partial r}ri-r\rho\frac{\partial\theta}{\partial r}=\frac{\partial\rho}{\partial\varphi}+ir\frac{\partial\theta}{\partial\varphi}$$

$$\begin{cases} r\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial r}=\dfrac{\partial\theta}{\partial\varphi} \\ -r\dfrac{\partial\theta}{\partial r}=\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial\varphi} \end{cases} \qquad \begin{cases} r\dfrac{\partial\ln\rho}{\partial r}=\rho\dfrac{\partial\theta}{\partial\varphi} \\ -r\dfrac{\partial\theta}{\partial r}=\dfrac{\partial\ln\rho}{\partial\varphi} \end{cases}$$

$$\begin{cases} \dfrac{\partial\theta}{\partial r}=\sin\varphi \\ \dfrac{\partial\theta}{\partial\varphi}=1+r\cos\varphi \end{cases}$$

$$\begin{cases} \dfrac{r}{\rho}\dfrac{\partial\rho}{\partial r}=\dfrac{\partial\theta}{\partial\varphi} \\ \dfrac{1}{\rho}\dfrac{\partial\rho}{\partial\varphi}=-r\dfrac{\partial\theta}{\partial r} \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial r}\left(\ln r\right)=\dfrac{1}{r}\left(1+r\cos\varphi\right)=\dfrac{1}{r}+\cos\varphi \\ \dfrac{\partial}{\partial\varphi}\left(\ln r\right)=-r\sin\varphi \end{cases}$$

$$\ln\rho=\ln r+r\cos\varphi+C_1$$
$$\rho=Cre^{r\cos\varphi},\ C>0$$

$$w=C\rho e^{r\cos\varphi}e^{i(\varphi+r\sin\varphi)}=C\rho e^{i\varphi}e^{r(\cos\varphi+i\sin\varphi)}=Cze^{z},C>0$$

4) 174 Найдите гармонические функции вида $u=\varphi\left(\dfrac{x^2+y^2}{x}\right)$.

$$\frac{\partial u}{\partial x} = \varphi' \cdot \left(1 - \frac{y^2}{x^2}\right), \; \frac{\partial^2 u}{\partial x^2} = \varphi'' \cdot \left(1 - \frac{y^2}{x^2}\right)^2 + \varphi' \cdot \frac{2y^2}{x^3};$$

$$\frac{\partial u}{\partial y} = \varphi' \cdot \frac{2y}{x}, \; \frac{\partial^2 u}{\partial y^2} = \varphi'' \cdot \frac{4y^2}{x^2} + \varphi' \cdot \frac{2}{x};$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \varphi'' \cdot \left(1 - \frac{y^2}{x^2}\right)^2 + \varphi' \cdot \frac{2y^2}{x^3} + \varphi'' \cdot \frac{4y^2}{x^2} + \varphi' \cdot \frac{2}{x} =$$

$$= \varphi'' \cdot \frac{1}{x^2}\left(x + \frac{y^2}{x}\right)^2 + \varphi' \cdot \frac{2}{x^2}\left(x + \frac{y^2}{x}\right) = \frac{1}{x^2}\left(\varphi'' \cdot \left(x + \frac{y^2}{x}\right)^2 + 2\varphi' \cdot \left(x + \frac{y^2}{x}\right)\right)$$

$$\varphi'' \cdot t^2 + 2\varphi' \cdot t = 0 \;\; \left(t = x + \frac{y^2}{x} = \frac{x^2 + y^2}{x}\right)$$

$$\varphi' \cdot t^2 = C_1, \; \varphi' = \frac{C_1}{t^2}, \; \varphi = \frac{C_1}{t} + C_2, \; C_1, C_2 \in \mathbb{R}$$

5) 186 Найдите аналитические функции, у которых вдоль любой линии семейства $x^2 + y^2 = Cx$ сохраняет постоянное значение либо вещественная часть, либо мнимая часть, либо модуль, либо аргумент.

1) $u = \dfrac{C_1 x}{x^2 + y^2} + C_2, v = -\dfrac{C_1 y}{x^2 + y^2} + C_3, w = \dfrac{C_1}{z} + \lambda, \; C_1 \in R, \lambda \in \mathbb{C}$

2) $v = \dfrac{C_1 x}{x^2 + y^2} + C_2, \; w = \dfrac{Ci}{z} + \lambda, \; C \in R, \lambda \in \mathbb{C}$

3) $\ln \rho = \dfrac{C_1 x}{x^2 + y^2} + C_2, \ln w = \dfrac{C}{z} + \lambda, w = \mu e^{C/z}, C \in \mathbb{R}, \mu \in \mathbb{C}$

4)

$$\theta = \frac{C_1 x}{x^2 + y^2} + C_2, \ln \rho = \frac{C_1 y}{x^2 + y^2} + C_3, \rho = e^{\frac{C_1 y}{x^2+y^2} + C_3}$$

$$w = \rho e^{i\theta} = e^{\frac{C_1 y}{x^2+y^2} + C_3 + i\left(\frac{C_1 x}{x^2+y^2} + C_2\right)} = e^{\frac{iC_1 x + C_1 y}{x^2+y^2} + C_3 + iC_2} = e^{\frac{iC_1(x - iy)}{x^2+y^2} + \lambda} = \mu e^{iC_1/x}$$