1) $\int_{-\pi}^{\pi} \frac{d\varphi}{5+3\cos\varphi} = \frac{\pi}{2}$

$z = e^{i\varphi},\ dz = e^{i\varphi} i d\varphi = iz d\varphi$

$$I = \int_{C:|z|=1} \frac{1}{5+3\frac{z+z^{-1}}{2}} \frac{dz}{iz} = \frac{2}{i} \int_{C:|z|=1} \frac{1}{3z^2+10z+3} dz$$

$z_1 = -1/3,\ z_2 = -3$

$$I = 4\pi \operatorname{Res}\left(\frac{1}{3(z-z_1)(z-z_2)}, z_1 \right) = 4\pi \frac{1}{3(z_1-z_2)} = \frac{4\pi}{8} = \frac{\pi}{2}$$

2) $\int_{-\pi}^{\pi} \frac{\cos^2\varphi d\varphi}{13+12\sin\varphi} = \frac{\pi}{9}$

$$I = \frac{1}{4} \int_{C:|z|=1} \frac{\left(z+z^{-1}\right)^2}{13+12\frac{z-z^{-1}}{2i}} \frac{dz}{iz} = \frac{1}{4} \int_{C:|z|=1} \frac{\left(z^2+1\right)^2}{6z^2+13iz-6} \frac{dz}{z^2}$$

$z_0 = 0,\ z_1 = -\frac{2}{3}i,\ z_2 = -\frac{3}{2}i$

$$\operatorname{Res}\left(\frac{\left(z^2+1\right)^2}{6z^2+13iz-6} \frac{1}{z^2}, 0 \right) = \frac{d}{dz}\left(\frac{\left(z^2+1\right)^2}{6z^2+13iz-6} \right)\Bigg|_{z=0} =$$

$$= \left(\frac{4z\left(z^2+1\right)}{6z^2+13iz-6} - \frac{\left(z^2+1\right)^2 (12z+13i)}{\left(6z^2+13iz-6\right)^2} \right)\Bigg|_{z=0} = -\frac{13i}{36}$$

$$\operatorname{Res}\left(\frac{\left(z^2+1\right)^2}{6z^2+13iz-6} \frac{1}{z^2}, z_1 \right) = \left(\frac{\left(z^2+1\right)^2}{6(z-z_2)} \frac{1}{z^2} \right)\Bigg|_{z=z_1} = \frac{\left(z_1^2+1\right)^2}{6(z_1-z_2)} \frac{1}{z_1^2} = \frac{1}{5i}\left(\frac{5}{9}\right)^2\left(-\frac{9}{4}\right) = \frac{5i}{36}$$

$$I = \frac{\pi i}{2}\left(\frac{-13i}{36} + \frac{5i}{36} \right) = \frac{8\pi}{72} = \frac{\pi}{9}$$

3) $\int\limits_0^{2\pi} \frac{\cos^4 \varphi d\varphi}{2+\sin^2 \varphi} = \left(3\sqrt{6}-7\right)\pi$

$$I = \int\limits_0^{2\pi} \frac{\cos^4 \varphi d\varphi}{2+\sin^2 \varphi} = \int\limits_0^{2\pi} \frac{\cos^4 \varphi d\varphi}{3-\cos^2 \varphi} d\varphi = \int\limits_0^{2\pi} \left(-\cos^2 \varphi - 3 + \frac{9}{3-\cos^2 \varphi}\right) d\varphi =$$

$$= -7\pi + 9\int\limits_0^{2\pi} \frac{1}{3-\cos^2 \varphi} d\varphi$$

$$\int\limits_0^{2\pi} \frac{1}{3-\cos^2 \varphi} d\varphi = \int\limits_{|z|=1} \frac{1}{3-\left(\frac{z+z^{-1}}{2}\right)^2} \frac{dz}{iz} = -\frac{4}{i} \int\limits_{|z|=1} \frac{z}{z^4-10z^2+1} dz =$$

$$z^2 = 5 \pm 2\sqrt{6}, \;\; z_{12} = \pm\sqrt{5-2\sqrt{6}}$$

$$-8\pi\left(\operatorname{Res}\left(\frac{z}{z^4-10z^2+1}, z_1\right) + \operatorname{Res}\left(\frac{z}{z^4-10z^2+1}, z_2\right)\right) =$$

$$= -8\pi\left(\frac{1}{4z_1^2-20} + \frac{1}{4z_2^2-20}\right) = -4\pi \frac{1}{5-2\sqrt{6}-5} = \sqrt{\frac{2}{3}}\pi$$

Ответ: $\left(3\sqrt{6}-7\right)\pi$

4) $\int_0^{\pi}\frac{\sin^2\varphi d\varphi}{1-2a\cos\varphi+a^2}=\frac{\pi}{2},\ \ 0<a<1$

$$I=\int_0^{2\pi}\frac{\sin^2\varphi d\varphi}{1-2a\cos\varphi+a^2}=-\frac{1}{4}\int_{C:|z|=1}\frac{\left(z-z^{-1}\right)^2}{1-a\left(z+z^{-1}\right)+a^2}\frac{dz}{iz}=$$

$$=-\frac{1}{4i}\int_{C:|z|=1}\frac{\left(z^2-1\right)^2}{\left(1+a^2\right)z-a\left(z^2+1\right)+a^2}\frac{dz}{z^2}=\frac{1}{4i}\int_{C:|z|=1}\frac{\left(z^2-1\right)^2}{az^2-\left(1+a^2\right)z+a}\frac{dz}{z^2};$$

$$f\left(z\right)=\frac{\left(z^2-1\right)^2}{az^2-\left(1+a^2\right)z+a}\frac{1}{z^2};\quad z_0=0,\ z_1=a,\ z_2=1/a.$$

$$\operatorname{Res}\left(f,\,0\right)=\frac{d}{dz}\left(\frac{\left(z^2-1\right)^2}{az^2-\left(1+a^2\right)z+a}\right)\Bigg|_{z=0}=\left(\frac{4z\left(z^2-1\right)}{az^2-\left(1+a^2\right)z+a}-\frac{\left(z^2-1\right)^2\left(2az-\left(1+a^2\right)\right)}{\left(az^2-\left(1+a^2\right)z+a\right)^2}\right)\Bigg|_{z=0}=\frac{a^2+1}{a^2}$$

$$\operatorname{Res}\left(f,\,a\right)=\frac{\left(z^2-1\right)^2}{a\left(z-\frac{1}{a}\right)}\frac{1}{z^2}\Bigg|_{z=a}=\frac{\left(a^2-1\right)^2}{\left(a^2-1\right)a^2}=\frac{a^2-1}{a^2}$$

$$\int_C f\left(z\right)dz=2\pi i\cdot 2=4\pi i$$

$$I=\int_0^{\pi}\frac{\sin^2\varphi d\varphi}{1-2a\cos\varphi+a^2}=\frac{\pi}{2},\ \ 0<a<1$$

5) $\int\limits_0^{\pi} \frac{\cos^2 \varphi d\varphi}{1 - a\sin^2 \varphi} = \frac{\pi}{a}\left(1 - \sqrt{1-a}\right),\ 0 < a < 1$

$$I_1 = \int\limits_0^{2\pi} \frac{\cos^2 \varphi d\varphi}{1 - a\sin^2 \varphi} = \frac{1}{4}\int\limits_{|z|=1} \frac{\left(z + z^{-1}\right)^2}{1 + a\frac{\left(z - z^{-1}\right)^2}{4}} \frac{dz}{iz} = \frac{1}{i}\int\limits_{|z|=1} \frac{\left(z^2 + 1\right)^2}{4z^2 + a\left(z^2 - 1\right)^2} \frac{dz}{z}.$$

$$f(z) = \frac{\left(z^2 + 1\right)^2}{4z^2 + a\left(z^2 - 1\right)^2} \frac{1}{z}.$$

;

$$az^4 - 2(a-2)z^2 + a = 0,\ z^2 = \frac{(a-2) \pm 2\sqrt{1-a}}{a},$$

$$z_1 = i\sqrt{\frac{2 - a - 2\sqrt{1-a}}{a}},\ z_2 = -i\sqrt{\frac{2 - a - 2\sqrt{1-a}}{a}};$$

$$z_3 = i\sqrt{\frac{2 - a + 2\sqrt{1-a}}{a}},\ z_4 = -i\sqrt{\frac{2 - a + 2\sqrt{1-a}}{a}}.$$

$$\operatorname{Res}(f,\ 0) = \frac{1}{a};$$

$$\operatorname{Res}(f,\ z_1) = \frac{\left(z_1^2 + 1\right)^2}{a\left(z_1^2 - z_3^2\right)\left(z_1 - z_2\right)} \frac{1}{z_1} = \frac{\left(z_1^2 + 1\right)^2}{a\left(z_1^2 - z_3^2\right)2z_1^2} = \frac{\left(z_1^2 + 1\right)^2}{a\left(z_1^2 - z_3^2\right)2z_1^2} = \frac{\left(z_1 + z_1^{-1}\right)^2}{2a\left(z_1^2 - z_3^2\right)} = \frac{z_1^2 + 2 + z_1^{-2}}{2a\left(z_1^2 - z_3^2\right)} =$$

$$= \frac{z_1^2 + 2 + z_3^2}{2a\left(z_1^2 - z_3^2\right)} = \frac{\frac{2(a-2)}{a} + 2}{2a\frac{4\sqrt{1-a}}{a}} = \frac{\frac{(a-2)}{a} + 1}{4a\frac{\sqrt{1-a}}{a}} = \frac{2a - 2}{4a\sqrt{1-a}} = \frac{a - 1}{2a\sqrt{1-a}} = -\frac{\sqrt{1-a}}{2a};$$

$$I_1 = 2\pi\left(\operatorname{Res}(f,\ 0) + \operatorname{Res}(f,\ z_1) + \operatorname{Res}(f,\ z_2)\right) =$$

$$= 2\pi\left(\frac{1}{a} - \frac{\sqrt{1-a}}{a}\right).$$

6) $\int\limits_{-\infty}^{+\infty} \frac{x^2-3x+4}{x^4+10x^2+9}dx = \frac{7}{12}\pi$

$$I = \int\limits_{-\infty}^{+\infty} \frac{x^2-3x+4}{x^4+10x^2+9}dx = \frac{7}{12}\pi;$$

$$f(z) = \frac{z^2-3z+4}{z^4+10z^2+9} = \frac{z^2-3z+4}{(z^2+1)(z^2+9)}$$

$$\int\limits_{C_R} \frac{z^2-3z+4}{z^4+10z^2+9}dz = 2\pi i\left(\operatorname{Res}(f,i)+\operatorname{Res}(f,3i)\right) = 2\pi i\left(\left.\frac{z^2-3z+4}{(z^2+9)(z+i)}\right|_{z=i} + \left.\frac{z^2-3z+4}{(z^2+1)(z+3i)}\right|_{z=3i}\right) =$$

$$= 2\pi i\left(\frac{3-3i}{8\cdot 2i}+\frac{-5-9i}{-8\cdot 6i}\right) = 2\pi\left(\frac{3}{8\cdot 2}+\frac{-5}{-8\cdot 6}\right) = \frac{7\pi}{12}$$

7) $\int\limits_{0}^{+\infty} \frac{x^4+1}{x^6+1}dx = \frac{2}{3}\pi$

$$f(z) = \frac{z^4+1}{z^6+1},\ \ z_k = e^{\left(\frac{\pi}{6}+\frac{\pi k}{3}\right)i};$$

$$\operatorname{Res}(f,z_k) = \frac{z_k^4+1}{6z_k^5};$$

$$\int\limits_{C_r} f(z)dz = 2\pi i\left(\operatorname{Res}(f,z_1)+\operatorname{Res}(f,z_2)+\operatorname{Res}(f,z_3)\right) =$$

$$= 2\pi i\left(\frac{e^{\frac{2\pi i}{3}}+1}{6e^{\frac{5\pi i}{6}}}+\frac{2}{6i}+\frac{e^{\frac{5\pi i}{3}}+1}{6e^{\frac{\pi i}{6}}}\right) = \frac{\pi i}{3}\left(\frac{e^{\frac{2\pi i}{3}}+1}{e^{\frac{5\pi i}{6}}}+\frac{2}{i}+\frac{e^{\frac{4\pi i}{3}}+1}{e^{\frac{\pi i}{6}}}\right) = \frac{\pi i}{3}\left(\frac{e^{\frac{\pi i}{3}}\left(e^{\frac{\pi i}{3}}+e^{-\frac{\pi i}{3}}\right)}{e^{\frac{5\pi i}{6}}}+\frac{2}{i}+\frac{e^{\frac{2\pi i}{3}}\left(e^{\frac{2\pi i}{3}}+e^{-\frac{2\pi i}{3}}\right)}{e^{\frac{\pi i}{6}}}\right) =$$

$$= \frac{\pi i}{3}(-i-2i-i) = \frac{4\pi}{3};$$

$$R\to+\infty:\quad \int\limits_{-\infty}^{+\infty}\frac{x^4+1}{x^6+1}dx = \frac{4}{3}\pi; \int\limits_{0}^{+\infty}\frac{x^4+1}{x^6+1}dx = \frac{2}{3}\pi;$$

8) $I=\int\limits_{0}^{+\infty}\frac{1}{\left(x^6+1\right)^2}dx=\frac{5}{18}\pi;$

$$f\left(z\right)=\frac{1}{\left(z^6+1\right)^2};\quad z_k=e^{\left(\frac{\pi}{6}+\frac{\pi k}{3}\right)i};$$

$$\operatorname{Res}\left(f,z_k\right)=\lim_{z\to z_k}\frac{d}{dz}\left(\frac{z-z_k}{z^6+1}\right)^2=\lim_{z\to z_k}2\left(\frac{z-z_k}{z^6+1}\right)\frac{z^6+1-6z^5\left(z-z_k\right)}{\left(z^6+1\right)^2}=$$

$$=\frac{1}{3z_k^5}\lim_{z\to z_k}\frac{-30z^4\left(z-z_k\right)}{12\left(z^6+1\right)z^5}=-\frac{30}{36z_k^6}\frac{1}{6z_k^5}=-\frac{5}{36}z_k$$

$$\int\limits_{C_R}f\left(z\right)dz=2\pi i\left(\operatorname{Res}\left(f,z_1\right)+\operatorname{Res}\left(f,z_2\right)+\operatorname{Res}\left(f,z_3\right)\right)=$$

$$-\frac{5\pi i}{18}\left(z_1+z_2+z_3\right)=-\frac{5\pi i}{18}\left(\frac{\sqrt{3}}{2}+\frac{i}{2}+i-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)=-\frac{5\pi i}{18}2i=\frac{5\pi}{9};$$

$$R\to+\infty:\ \int\limits_{-\infty}^{+\infty}\frac{1}{\left(x^6+1\right)^2}dx=\frac{5}{9}\pi;\quad \int\limits_{0}^{+\infty}\frac{1}{\left(x^6+1\right)^2}dx=\frac{5}{18}\pi;$$

9) $\int\limits_{0}^{+\infty} \frac{x^2}{\left(x^4+4\right)^3}\,dx = \frac{5}{2048}\pi$

$$f\left(z\right)=\frac{z^2}{\left(z^4+4\right)^3}, \quad z_k=\sqrt{2}e^{\left(\frac{\pi}{4}+\frac{\pi k}{2}\right)i};$$

$$\operatorname{Res}\left(f,z_k\right)=\lim_{z\to z_k}\frac{1}{2}\frac{d^2}{dz^2}\left(z^2\left(\frac{z-z_k}{z^4+4}\right)^3\right)=$$

$$=\lim_{z\to z_k}\frac{1}{2}\frac{d}{dz}\left(2z\left(\frac{z-z_k}{z^4+4}\right)^3+3z^2\left(\frac{z-z_k}{z^4+4}\right)^2\frac{d}{dz}\left(\frac{z-z_k}{z^4+4}\right)\right)=$$

$$=\lim_{z\to z_k}\frac{1}{2}\left(2\left(\frac{z-z_k}{z^4+4}\right)^3+12z\left(\frac{z-z_k}{z^4+4}\right)^2\frac{d}{dz}\left(\frac{z-z_k}{z^4+4}\right)+6z^2\left(\frac{z-z_k}{z^4+4}\right)\left(\frac{d}{dz}\left(\frac{z-z_k}{z^4+4}\right)\right)^2+3z^2\left(\frac{z-z_k}{z^4+4}\right)^2\frac{d^2}{dz^2}\left(\frac{z-z_k}{z^4+4}\right)\right);$$

$$\lim_{z\to z_k}\frac{z-z_k}{z^4+4}=\frac{1}{4z_k^3};$$

$$\frac{d}{dz}\left(\frac{z-z_k}{z^4+4}\right)=\frac{z^4+4-4z^3\left(z-z_k\right)}{\left(z^4+4\right)^2};$$

$$\lim_{z\to z_k}\frac{d}{dz}\left(\frac{z-z_k}{z^4+4}\right)=\lim_{z\to z_k}\frac{-12z^2\left(z-z_k\right)}{8z^3\left(z^4+4\right)}=-\frac{3}{2z_k}\frac{1}{4z_k^3}=-\frac{3}{8z_k^4}=\frac{3}{32};$$

$$\frac{d^2}{dz^2}\left(\frac{z-z_k}{z^4+4}\right)=\frac{-12z^2(z-z_k)(z^4+4)-8z^3\left(z^4+4-4z^3(z-z_k)\right)}{(z^4+4)^3};$$

$$\lim_{z\to z_k}\frac{d^2}{dz^2}\left(\frac{z-z_k}{z^4+4}\right)=$$

$$=\lim_{z\to z_k}\frac{-24z(z-z_k)(z^4+4)-12z^2(z^4+4)-48z^5(z-z_k)-24z^2\left(z^4+4-4z^3(z-z_k)\right)-8z^3\left(-12z^2(z-z_k)\right)}{12z^3(z^4+4)^2}=$$

$$=\lim_{z\to z_k}\frac{-24z(z-z_k)(z^4+4)-36z^2(z^4+4)+144z^5(z-z_k)}{12z^3(z^4+4)^2}=$$

$$=\frac{1}{z_k^2}\lim_{z\to z_k}\frac{-2(z-z_k)(z^4+4)-3z(z^4+4)+12z^4(z-z_k)}{(z^4+4)^2}=$$

$$=\frac{1}{z_k^2}\lim_{z\to z_k}\frac{-2(z^4+4)-8z^3(z-z_k)-3(z^4+4)-12z^4+48z^3(z-z_k)+12z^4}{8z^3(z^4+4)}=$$

$$=\frac{1}{z_k^2}\lim_{z\to z_k}\frac{-5(z^4+4)+40z^3(z-z_k)}{8z^3(z^4+4)}=\frac{1}{8z_k^5}\left(-5+40z_k^3\frac{1}{4z_k^3}\right)=\frac{5}{8z_k^5}=-\frac{5}{32z_k}$$

$$\operatorname{Res}(f,z_k)=\lim_{z\to z_k}\frac{1}{2}\frac{d^2}{dz^2}\left(z^2\left(\frac{z-z_k}{z^4+4}\right)^3\right)=$$

$$\frac{1}{2}\left(2\left(\frac{1}{4z_k^3}\right)^3+12z_k\left(\frac{1}{4z_k^3}\right)^2\frac{3}{32}+6z_k^2\left(\frac{1}{4z_k^3}\right)\left(\frac{3}{32}\right)^2+3z_k^2\left(\frac{1}{4z_k^3}\right)^2\frac{-5}{32z_k}\right)=$$

$$=\frac{1}{2}\left(2\left(\frac{1}{4z_k^3}\right)^3+21z_k\left(\frac{1}{4z_k^3}\right)^2\frac{1}{32}+6z_k^2\left(\frac{1}{4z_k^3}\right)\left(\frac{3}{32}\right)^2\right)=\frac{1}{2}\left(2\frac{1}{64z_k^9}+21\frac{1}{16z_k^5}\frac{1}{32}+6z_k^2\left(\frac{1}{4z_k^3}\right)\left(\frac{3}{32}\right)^2\right)=$$

$$=\frac{1}{2}\left(2\frac{1}{64\cdot16z_k}-21\frac{1}{64z_k}\frac{1}{32}+6\left(\frac{1}{4z_k}\right)\left(\frac{3}{32}\right)^2\right)=\frac{1}{2z_k}\left(\frac{1}{512}-\frac{21}{2048}+\frac{27}{2048}\right)=\frac{1}{2z_k}\left(\frac{1}{512}+\frac{3}{1024}\right)=\frac{5}{2048z_k};$$

$$\int_{C_r}\frac{z^2}{(z^4+4)^3}dz=2\pi i\left(\operatorname{Res}(f,z_1)+\operatorname{Res}(f,z_2)\right)=\frac{5\pi i}{1024}\left(\frac{1}{z_1}+\frac{1}{z_2}\right)=\frac{5\pi i}{1024}\left(\frac{1}{1+i}+\frac{1}{-1+i}\right)=\frac{5\pi i}{2048}(-2i)=\frac{5\pi}{1024};$$

$$R\to+\infty:\quad \int_{-\infty}^{+\infty}\frac{x^2}{(x^4+4)^3}dx=\frac{5}{1024}\pi,\quad \int_{0}^{+\infty}\frac{x^2}{(x^4+4)^3}dx=\frac{5}{2048}\pi$$

10) $\int\limits_0^{+\infty} \frac{x^2}{\left(x^4+4\right)\left(x^2+9\right)^2}\,dx = \frac{\pi}{867}\pi$

$$f\left(z\right)=\frac{z^2}{\left(z^4+4\right)\left(z^2+9\right)^2}, \qquad z_1=1+i,\ z_2=-1+i,\ z_2=3i;$$

$$\operatorname{Res}\left(f,z_1\right)=\left.\frac{z^2}{4z^3\left(z^2+9\right)^2}\right|_{z=1+i}=\left.\frac{1}{4z\left(z^2+9\right)^2}\right|_{z=1+i}=\frac{1}{4\left(1+i\right)\cdot\left(9+2i\right)^2}=$$

$$=\frac{1}{4\left(1+i\right)\left(77+36i\right)}=\frac{1}{4\cdot\left(41+113i\right)}=\frac{41-113i}{4\cdot 14450};$$

$$\operatorname{Res}\left(f,z_2\right)=\left.\frac{z^2}{4z^3\left(z^2+9\right)^2}\right|_{z=-1+i}=\left.\frac{1}{4z\left(z^2+9\right)^2}\right|_{z=-1+i}=\frac{1}{4\left(-1+i\right)\cdot\left(9-2i\right)^2}=$$

$$=\frac{1}{4\left(-1+i\right)\left(77-36i\right)}=\frac{1}{4\cdot\left(-41+113i\right)}=\frac{-41-113i}{4\cdot 14450};$$

$$\operatorname{Res}\left(f,z_3\right)=\left.\frac{d}{dz}\left(\frac{z^2}{\left(z^4+4\right)\left(z+3i\right)^2}\right)\right|_{z=3i}=\left.\left(\frac{2z}{\left(z^4+4\right)\left(z+3i\right)^2}-\frac{4z^5}{\left(z^4+4\right)^2\left(z+3i\right)^2}-\frac{2z^2}{\left(z^4+4\right)\left(z+3i\right)^3}\right)\right|_{z=3i}=$$

$$=\left(\frac{6i}{85\cdot\left(-36\right)}-\frac{4\cdot 243i}{7225\cdot\left(-36\right)}-\frac{-18}{85\cdot\left(-216i\right)}\right)=-\frac{i}{85\cdot 6}+\frac{2\cdot 81i}{7225\cdot 6}+\frac{i}{85\cdot 12}=\frac{-170+324+85}{7225\cdot 12}i=\frac{239}{7225\cdot 12}i;$$

$$\int\limits_{C_R} f\left(z\right)dz=2\pi i\left(\frac{41-113i}{4\cdot 14450}+\frac{-41-113i}{4\cdot 14450}+\frac{239}{7225\cdot 12}i\right)=2\pi\left(\frac{113}{2\cdot 14450}-\frac{239}{7225\cdot 12}\right)=$$

$$=2\pi\left(\frac{113}{2\cdot 50\cdot 17^2}-\frac{239}{17^2\cdot 25\cdot 12}\right)=2\pi\frac{339-239}{17^2\cdot 300}=\frac{2\pi}{289\cdot 3}=\frac{2\pi}{867}$$

$$R\to+\infty:\quad \int\limits_0^{+\infty}\frac{x^2}{\left(x^4+4\right)\left(x^2+9\right)^2}\,dx=\frac{\pi}{867} \qquad \int\limits_0^{+\infty}\frac{x^2}{\left(x^4+4\right)\left(x^2+9\right)^2}\,dx=\frac{\pi}{867}$$