Занятие 10.

Вычеты. Интегралы по замкнутым контурам

Вычислите вычеты во всех изолированных особых точках, включая ∞; убедитесь, что сумма вычетов равна нулю.

621

$$\frac{1}{z^3 - z^5}$$

$$\operatorname{Res}\left(\frac{1}{z^3 - z^5},\, 0\right) = \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{1}{1-z^2}\right)\Bigg|_{z=0} = \frac{1}{4}\frac{d^2}{dz^2}\left(\frac{1}{1-z} + \frac{1}{1+z}\right)\Bigg|_{z=0} =$$

$$= \frac{1}{2}\left(\frac{1}{(1-z)^3} + \frac{1}{(1+z)^3}\right)\Bigg|_{z=0} = 1$$

$$\operatorname{Res}\left(\frac{1}{z^3 - z^5},\, 1\right) = \left(-\frac{1}{z^3(1+z)}\right)\Bigg|_{z=1} = -\frac{1}{2};$$

$$\operatorname{Res}\left(\frac{1}{z^3 - z^5},\, -1\right) = \left(\frac{1}{z^3(1-z)}\right)\Bigg|_{z=-1} = -\frac{1}{2};$$

$$\operatorname{Res}\left(\frac{1}{z^3 - z^5},\, \infty\right) = 0$$

625

$$\frac{z^2 + z - 1}{z^2(z-1)}$$

$$\operatorname{Res}\left(\frac{z^2 + z - 1}{z^2(z-1)},\, 0\right) = \frac{d}{dz}\left(\frac{z^2 + z - 1}{z - 1}\right)\Bigg|_{z=0} = \frac{(2z+1)(z-1) - (z^2 + z - 1)}{(z-1)^2}\Bigg|_{z=0} = 0$$

или

$$\frac{z^2 + z - 1}{z^2(z-1)} = -\left(1 + \frac{1}{z} - \frac{1}{z^2}\right)\frac{1}{1-z} = -\left(1 + \frac{1}{z} - \frac{1}{z^2}\right)\left(1 + z + z^2 + \cdots\right)$$

$$\operatorname{Res}\left(\frac{z^2 + z - 1}{z^2(z-1)},\, 0\right) = -1 + 1 = 0$$

$$\operatorname{Res}\left(\frac{z^2+z-1}{z^2(z-1)},1\right)=\left.\frac{z^2+z-1}{z^2}\right|_{z=1}=1$$

$$\operatorname{Res}\left(\frac{z^2+z-1}{z^2(z-1)},\infty\right)=-1$$

$$\frac{z^2+z-1}{z^2(z-1)}=\left(1+\frac{1}{z}-\frac{1}{z^2}\right)\frac{1}{z}\frac{1}{1-\frac{1}{z}}=\left(1+\frac{1}{z}-\frac{1}{z^2}\right)\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

627

$$\frac{e^z}{z^2(z^2+9)}$$

$$\operatorname{Res}\left(\frac{e^z}{z^2(z^2+9)},0\right)=\left.\frac{d}{dz}\left(\frac{e^z}{z^2+9}\right)\right|_{z=0}=\left.\frac{e^z(z^2+9)-e^z2z}{(z^2+9)^2}\right|_{z=0}=\frac{1}{9}$$

$$\operatorname{Res}\left(\frac{e^z}{z^2(z^2+9)},3i\right)=\left.\frac{e^z}{z^2(z+3i)}\right|_{z=3i}=\frac{e^{3i}}{-9\cdot 6i}=\frac{i}{54}e^{3i};$$

$$\operatorname{Res}\left(\frac{e^z}{z^2(z^2+9)},-3i\right)=\left.\frac{e^z}{z^2(z-3i)}\right|_{z=3i}=\frac{e^{-3i}}{-9\cdot(-6i)}=-\frac{i}{54}e^{-3i}.$$

$$\frac{e^z}{z^2(z^2+9)}=\frac{1}{z^4}\frac{1}{1+\frac{9}{z^2}}e^z=\frac{1}{z^4}\sum_{n=0}^{\infty}(-1)^n\frac{9^n}{z^{2n}}\sum_{m=0}^{\infty}\frac{z^m}{m!};$$

$$-4-2n+m=-1,\ \ m=2n+3,$$

$$c_{-1}=\sum_{n=0}^{\infty}(-1)^n\frac{9^n}{(2n+3)!}=\sum_{n=2}^{\infty}(-1)^n\frac{9^{n-2}}{(2n-1)!}=$$

$$-\frac{1}{27}\left(\sum_{n=2}^{\infty}(-1)^{n-1}\frac{3^{2n-1}}{(2n-1)!}\right)=-\frac{1}{27}\left(\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{2n-1}}{(2n-1)!}-3\right)=-\frac{1}{27}\sin 3+\frac{1}{9},$$

$$\operatorname{Res}\left(\frac{e^z}{z^2(z^2+9)},\infty\right)=\frac{1}{27}\sin 3-\frac{1}{9}$$

$$f(z)=\frac{e^{\frac{2}{z}}}{(z+3)}$$

$$\operatorname{Res}\left(\frac{e^{\frac{2}{z}}}{z+3},-3\right)=e^{\frac{2}{z}}\bigg|_{z=-3}=e^{\frac{-2}{3}},$$

$$\operatorname{Res}\left(\frac{e^{\frac{2}{z}}}{z+3},0\right):$$

$$\frac{e^{\frac{2}{z}}}{z+3}=\frac{1}{3}\frac{e^{\frac{2}{z}}}{1+\frac{z}{3}}=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{2^n}{z^n}\sum_{m=0}^{\infty}(-1)^m\frac{z^m}{3^{m+1}}$$

$$-n+m=-1,\ m=n-1,$$

$$c_{-1}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{3^n n!}=-e^{-\frac{2}{3}}+1$$

$$\operatorname{Res}\left(\frac{e^{\frac{2}{z}}}{z+3},0\right)=-e^{\frac{-2}{3}}+1$$

$$\operatorname{Res}\left(\frac{e^{\frac{2}{z}}}{z+3},\infty\right):$$

$$\frac{e^{\frac{2}{z}}}{z+3}=\frac{1}{z}\frac{e^{\frac{2}{z}}}{1+\frac{3}{z}}=\frac{1}{z}\left(1+\frac{2}{z}+\cdots\right)\left(1-\frac{3}{z}+\cdots\right),$$

$$c_{-1}=1.$$

$$\operatorname{Res}\left(\frac{e^{\frac{2}{z}}}{z+3},\infty\right)=-c_{-1}=-1.$$

$$f(z)=\frac{\cos 3z}{(z+4)^2}$$

$$\operatorname{Res}\left(\frac{\cos 3z}{(z+4)^2},-4\right)=\left(\frac{d}{dz}(\cos 3z)\right)\Bigg|_{z=-4}=(-3\sin 3z)\big|_{z=-4}=3\sin 12.$$

$$\operatorname{Res}\left(\frac{\cos 3z}{(z+4)^2},\infty\right):$$

$$\frac{\cos 3z}{(z+4)^2}=\frac{1}{z^2}\frac{\cos 3z}{\left(1+\frac{4}{z}\right)^2}=\frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\frac{3^{2n}z^{2n}}{(2n)!}\sum_{m=0}^{\infty}(-1)^m(m+1)\frac{4^m}{z^m},$$

$$-2+2n-m=-1,\ \ m=2n-1,$$

$$c_{-1}=-\sum_{n=1}^{\infty}(-1)^n 2n\frac{3^{2n}4^{2n-1}}{(2n)!}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{2n}4^{2n-1}}{(2n-1)!}=3\sin 12$$

$$\operatorname{Res}\left(\frac{\cos 3z}{(z+4)^2},\infty\right)=-3\sin 12$$

Вычислите интегралы.

657

$$\int_C\frac{dz}{z^4+1},\ \ C: x^2+y^2=2x$$

$$I=\int_C\frac{dz}{z^4+1},\ \ C: x^2+y^2=2x$$

$$z_{1,2}=e^{\pm\frac{\pi i}{4}},$$

$$\operatorname{Res}\left(\frac{1}{z^4+1},z_k\right)=\frac{1}{4z_k^3};$$

$$I=2\pi i\frac{1}{4}\left(\frac{1}{z_1^3}+\frac{1}{z_2^3}\right)=\frac{\pi i}{2}\left(e^{\frac{3\pi i}{4}}+e^{\frac{3\pi i}{4}}\right)=\pi i\cos\frac{3\pi}{4}=-\frac{\pi i}{\sqrt{2}}$$