Лемма Жордана.

Пусть функция f непрерывна на $\operatorname{Im} z \geq 0,\ |z| \geq R_0$,

$$M_R = \max_{|z|=R,\ \operatorname{Im} z \geq 0} |f(z)| \underset{R\to+\infty}{\longrightarrow} 0,$$

$\alpha > 0$.

Тогда

$$I_R = \int\limits_{C_R : |z|=R, \operatorname{Im} z \geq 0} f(z) e^{i\alpha z} dz \underset{R\to\infty}{\longrightarrow} 0.$$

Лемма

f имеет простой полюс в нуле. C_r — верхняя полуокружность радиуса r с центром в 0.

Тогда $\int\limits_{C_r} f(z)\,dz \underset{r\to+0}{\longrightarrow} \pi i \operatorname{Res}(f, 0)$.

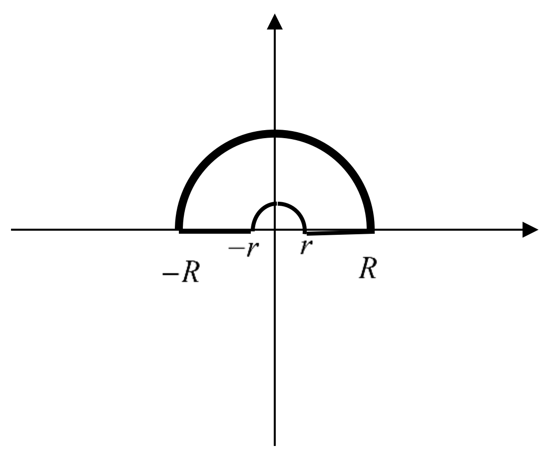
Доказательство. $f(z) = \dfrac{c_{-1}}{z} + g(z)$. g — голоморфная функция. g ограничена в окрестности нуля $|z| < r_0$:

$$\exists M > 0 : |g(z)| \leq M \ \text{при}\ |z| < r_0.$$

$$\left|\int\limits_{C_r} g(z)\,dz\right| \leq M\pi r \ \text{при}\ |z| < r_0;\ \left|\int\limits_{C_r} g(z)\,dz\right| \leq M\pi r \underset{r\to+0}{\longrightarrow} 0.$$

$$\int\limits_{C_r} \frac{dz}{z} = \left[z = e^{it}\right] = \int\limits_0^{\pi} \frac{e^{it}}{e^{it}} i\,dt = \pi i$$

1) $\displaystyle\int\limits_0^{+\infty} \frac{x - \sin x}{x^3} dx = \frac{\pi}{4}$



$$\int_{\Gamma_{rR}} \frac{e^{iz}-1-iz}{z^3}\,dz = 0$$

$$\operatorname{Res}\left(\frac{e^{iz}-1-iz}{z^3},0\right) = \lim_{z\to 0}\frac{e^{iz}-1-iz}{z^2} = -\frac{1}{2}$$

$$r\to 0, R\to +\infty$$

$$v.p.\int_{-\infty}^{+\infty}\frac{e^{ix}-1-ix}{x^3}\,dx + \frac{\pi i}{2} = 0,\quad \int_{-\infty}^{+\infty}\frac{\sin x - x}{x^3}\,dx + \frac{\pi}{2} = 0,\quad \int_{0}^{+\infty}\frac{x-\sin x}{x^3}\,dx = \frac{\pi}{4}$$

2)

$$\int_{0}^{+\infty}\frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$$

$$\int_{\Gamma_{rR}}\frac{e^{3iz}-3e^{iz}+2}{z^3}\,dz = 0\,.$$

$$\frac{e^{3iz}-3e^{iz}+2}{z^3} = \frac{1}{z^3}\left(\left(1+3iz-\frac{9}{2}z^2+\cdots\right)-3\left(1+iz-\frac{1}{2}z^2+\cdots\right)+2\right) = \frac{1}{z^3}\left(-3z^2+\cdots\right) = -\frac{3}{z}+\cdots$$

$$\operatorname{Res}\left(\frac{e^{3iz}-3e^{iz}+2}{z^3},0\right) = -3$$

$$v.p.\int_{-\infty}^{+\infty}\frac{e^{3ix}-3e^{ix}+2}{x^3}\,dx + 3\pi i = 0$$

$$\int_{-\infty}^{+\infty}\frac{\sin 3x - 3\sin x}{x^3}\,dx + 3\pi = 0$$

$$\int_{0}^{+\infty}\frac{\sin 3x - 3\sin x}{x^3}\,dx = -\frac{3\pi}{2}$$

$$\sin 3x - 3\sin x = \sin 2x\cos x + \sin x\cos 2x - 3\sin x = 2\sin x\cos^2 x + \sin x\left(1-2\sin^2 x\right) - 3\sin x =$$
$$= 2\sin x\left(1-\sin^2 x\right) + \sin x\left(1-2\sin^2 x\right) - 3\sin x = -4\sin^3 x$$

$$-4\int_{0}^{+\infty}\frac{\sin^3 x}{x^3}\,dx = -\frac{3\pi}{2}$$

$$\int_{0}^{+\infty}\frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$$

3)

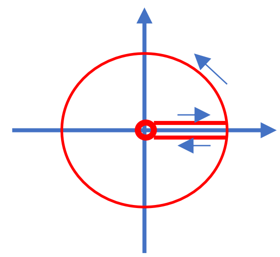
$$I=\int_0^{+\infty}\frac{dx}{x^3+1}=\frac{2\pi}{3\sqrt{3}}$$

А) $\operatorname{Res}\left(\frac{\ln z}{z^3+1},z_k\right)=\frac{\ln z_k}{3z_k^2},\ z_1=e^{\frac{\pi i}{3}},z_2=-1,z_3=e^{\frac{5\pi i}{3}}$

$$\int_{\Gamma_{rR}}\frac{\ln z}{z^3+1}dz=\frac{2\pi i}{3}\left(\frac{\pi i/3}{e^{2\pi i/3}}+\pi i+\frac{5\pi i/3}{e^{10\pi i/3}}\right)=\frac{2\pi i}{3}\left(\frac{\pi i}{3}e^{-2\pi i/3}+\pi i+\frac{5\pi i}{3}e^{-10\pi i/3}\right)=$$

$$=\frac{-2\pi^2}{9}\left(e^{-2\pi i/3}+3+5e^{2\pi i/3}\right)=\frac{-2\pi^2}{9}\left(\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)+3+5\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)=\frac{-2\pi^2}{9}\left(2\sqrt{3}i\right)=\frac{-4\pi^2 i}{3\sqrt{3}}$$

$$\int_0^{+\infty}\frac{\ln x}{x^3+1}dx-\int_0^{+\infty}\frac{\ln x+2\pi i}{x^3+1}dx=\frac{-4\pi^2 i}{3\sqrt{3}}$$



$$\int_0^{+\infty}\frac{1}{x^3+1}dx=\frac{2\pi}{3\sqrt{3}}$$

Б)

$$I\left(1-e^{2\pi i/3}\right)=2\pi i\operatorname{Res}\left(\frac{1}{3z^2}\right)\Bigg|_{e^{\pi i/3}}=\frac{2\pi i}{3e^{2\pi i/3}}=\frac{2}{3}\pi ie^{-2\pi i/3}$$

$$Ie^{\pi i/3}\left(e^{-\pi i/3}-e^{\pi i/3}\right)=\frac{2}{3}\pi ie^{-2\pi i/3}$$

$$-2i\sin\frac{\pi}{3}Ie^{\pi i/3}=\frac{2}{3}\pi ie^{-2\pi i/3}$$

$$I=\frac{2}{3\sqrt{3}}\pi$$